Degrees of Freedom of Interference Channel with Rank-Deficient Transfer Matrix

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***Abstract*—We consider the interference channel with K transmitters and K receivers all having a single antenna, wherein the $K \times K$ transfer matrix representing this channel has rank D $(D < K)$. The degrees of freedom (DoF) of such channels are not known as the rank deficiency in the transfer matrix creates algebraic dependencies between the channel coefficients. We present a modified version of the [CJ08] alignment scheme, to handle these dependencies while aligning interference, and state the sufficient conditions for achieving half rate per user using this scheme. The difficulties in proving these sufficient conditions are shown for $K = 4$ and $K = 5$. We also show that these sufficient conditions are not satisfied for $K \geq 6$.**

I. Introduction

Understanding the degrees of freedom (DoF) of interference networks is a significant problem in network information theory, which has motivated many fundamental ideas. Optimal DoF results are available for several K-user interference channels (SISO or MIMO) using the principle of Interference Alignment [1]–[4]. When all transmitters and receivers have M antennas with full rank channel matrices, it is known that $\frac{KM}{2}$ DoF are achievable using the [CJ08] asymptotic alignment scheme [1], if the channel coefficients are time-varying and drawn from a continuous distribution.

The DoF of rank deficient MIMO interference channels have been studied in [5]–[8]. All these prior works consider individual channels between a transmitter-receiver pair to be rank deficient. Such rank deficient channels are frequently encountered in wireless MIMO networks due to poor scattering and keyhole effects. This paper considers the overall transfer matrix of the network to be rank deficient, which has not been explored before. Rank deficient transfer matrices are observed typically in wired and wireless networks with constraints in the network topology. For example, such rank deficient transfer matrices could manifest in relay networks, wherein all the intelligence resides only at the transmitters. Rank deficiency in the transfer matrix leads to spatial dependencies between the direct and cross channels, implications of which will be discussed in this work.

The DoF of 2-user SISO interference channel with such rank deficiencies are known trivially, while those for 3-user SISO interference channel follows from [9]–[11]. The use of interference alignment for the 3 multiple unicast network coding problem was initially discussed in [9] and [10]. Later, Meng et al. derived the feasibility conditions for asymptotic interference alignment, in [11]. Rank deficiency in X channels was discussed in [12], with the individual channels being rank deficient. Spatial dependencies have also been observed in interference channels with coordinated multipoint (CoMP) transmission and reception, the DoF of which were explored in [13].

In this paper, we introduce the problem of characterizing the DoF of K-user SISO interference channels with transfer matrix of rank D $(D < K)$. We present a modified version of the asymptotic interference alignment [CJ08] scheme to handle the spatial dependencies that arise due to the rank deficiency. A set of polynomial conditions are derived which are shown to be sufficient for achieving half rate per user using this modified scheme. We analyze the 4-user and 5-user interference channels with rank D, and point out the difficulty in proving the sufficient conditions here. We then study the 6-user interference channel where we show that the sufficient conditions are not satisfied, thereby pointing out the challenges in showing achievability for $K \geq 6$.

II. System Model

We consider the K-user SISO interference channel with perfect global channel knowledge. The channel output at the k-th receiver over the t-th channel use is given as,

$$Y_k(t) = \sum_{j=1}^{K} H_{kj}(t)X_j(t) + Z_k(t)$$

where, $k \in \{1, 2, \ldots, K\}$ is the user index, $t \in \mathbb{N}$ is the channel use index, $Y_k(t)$ is the output signal of the k-th receiver, $X_k(t)$ is the input signal of the k-th transmitter, $H_{kj}(t)$ is the channel coefficient from transmitter j to receiver k over the t-th channel use, and $Z_k(t)$ is the AWGN at the k-th receiver. The bold face notations $\mathbf{X}_k, \mathbf{Y}_k$, and $\mathbf{Z}_k$ are used to represent the vector form of their corresponding scalars over multiple channel uses, and the bold face notation $\mathbf{H}_{ij}$ is used to represent the diagonal channel matrix over multiple channel uses. For any given time slot $t \in \mathbb{N}$, the overall transfer matrix is defined as the $K \times K$ matrix of the form $\mathbf{H}(t) = [H_{ij}(t)]\ \forall i, j \in \{1, 2, \ldots, K\}$, and its rank is given by D. Time indices are omitted for brevity.

Let $R_k(\rho)$ denote the achievable rate of user k where ρ is the Signal-to-Noise Ratio (SNR). The capacity region $C(\rho)$ of this network is the set of achievable rate tuples $R(\rho) = (R_1(\rho), R_2(\rho), \ldots R_K(\rho))$, such that each user can simultaneously decode its desired message with arbitrarily small error probability. The maximum sum rate of this channel is defined as $R_\Sigma(\rho) = max_{R(\rho)\in C(\rho)} \sum_{k=1}^{K} R_k(\rho)$. The sum

DoF is defined as $d_\Sigma = lim_{\rho \to \inf} \frac{R_\Sigma(\rho)}{log(\rho)}$ and $\frac{d_\Sigma}{K}$ as the normalized DoF per user.

III. Overview Of Results

For the K-user SISO interference channel with rank deficient transfer matrix $(D < K)$, we show that the outer bound of the sum degrees of freedom is : $d_\Sigma \leq \min\{D, \frac{K}{2}\}$.

The rank deficiency in the transfer matrix creates algebraic dependencies even among the cross channel coefficients making it hard to apply the [CJ08] alignment scheme directly. We introduce a modified version of the [CJ08] scheme to deal with these dependencies, and then derive the sufficient conditions under which there will be no overlap between the desired and interfering signal spaces. This scheme is also used in [14] to show achievability results for individual channel rank deficiency of MIMO interference channels.

Let $\mathbb{S}$ denote the set of channel realizations for which the rank of the transfer matrix does not exceed D.

Theorem 1: Degrees of freedom achievable for the K-user interference channel with rank deficient transfer matrix, can be made arbitrarily close to half per user using the modified alignment scheme, if for each $k \in \{1, \ldots, K\} : QH_{kk} - P \neq 0$ $\forall$ P, Q ; where P and Q are multivariate polynomials in the variables $\{H_{ij} : i \neq j\}$ and non-zero under $\mathbb{S}$.

This theorem signifies that half rate per user is achievable even with algebraic dependencies among the channels, provided the dependencies between the direct and cross channels can not be expressed in the form defined above.

We then check if these conditions hold true for the general K-user case. We discuss a simple approach that uses ergodic alignment [15] ideas to get a subspace of realizations which in turn would help us prove some, but not all, of these sufficient conditions.

We present our analysis for $K = 4, 5$ and explain the difficulties of proving the sufficient conditions from Theorem 1. We also study the 6-user channel, and show that the sufficient conditions are not satisfied.

IV. Mathematical Preliminaries

In the appendix, we use some results in algebraic geometry, so we start by recalling some basic terminology in algebraic geometry. We refer the reader to the book [16] for an excellent introduction.

A. Varities and Ideals

Let $\mathbb{C}[t_1, t_2, \cdots, t_n]$ and $\mathbb{C}(t_1, t_2, \cdots, t_n)$ denote the set of multivariate polynomials and rational functions, respectively, in the variables $t_1, t_2, \cdots, t_n$. For any polynomials $f_1, f_2, \cdots, f_m \in \mathbb{C}[t_1, t_2, \cdots, t_n]$, the *affine variety* generated by $f_1, f_2, \cdots, f_m$ is defined as set of points at which the polynomials vanish:

$$V(\mathbf{f}) = \{\mathbf{t} \in \mathbb{C}^n : \mathbf{f}(\mathbf{t}) = \mathbf{0}\}.$$

Any subset $I \subseteq \mathbb{C}[t_1, t_2, \cdots, t_n]$ is called an *ideal* if it satisfies the three properties

- $0 \in I$.
- If $f_1, f_2 \in I$, then $f_1 + f_2 \in I$.
- If $f_1 \in I$ and $f_2 \in \mathbb{C}[t_1, t_2, \cdots, t_n]$, then $f_1 f_2 \in I$.

For any set $\mathcal{A} \subseteq \mathbb{C}^n$, the ideal generated by $\mathcal{A}$ is defined as

$$I(\mathcal{A}) = \{f \in \mathbb{C}[t_1, t_2, \cdots, t_n] : f(\mathbf{t}) = 0 \ \forall \mathbf{t} \in \mathcal{A}\}.$$

For any ideal I, the affine variety generated by I is defined as

$$V(I) = \{\mathbf{t} \in \mathbb{C}^n : f(\mathbf{t}) = 0 \ \forall f \in I\}.$$

B. Algebraic Independence and Jacobian Criterion

Definition 1: The rational functions $f_1, f_2 \cdots, f_m \in \mathbb{C}(t_1, t_2, \cdots, t_n)$ are called algebraically dependent (over $\mathbb{C}$) if there exists a nonzero polynomial $F \in \mathbb{C}[s_1, s_2 \cdots, s_m]$ such that $F(f_1, f_2, \cdots, f_m) = 0$. If there exists no such annihilating polynomial F, then $f_1, f_2, \cdots, f_m$ are algebraically independent.

Lemma 1 (Theorem 3 on page 135 of [17]): The rational functions $f_1, f_2 \cdots, f_m \in \mathbb{C}(t_1, t_2, \cdots, t_n)$ are algebraically independent if and only if the Jacobian matrix

$$\mathbf{J}_f = \left(\frac{\partial f_i}{\partial t_j}\right)_{1 \leq i \leq m, 1 \leq j \leq n} \tag{1}$$

has full row rank equal to m.

The Jacobian matrix is a function of the variables $t_1, t_2, \cdots, t_n$, and hence the Jacobian matrix can have different ranks at different points $\mathbf{t} \in \mathbb{C}^n$. The above lemma refers to the *structural rank* of the Jacobian matrix which is equal to m if and only if there exists at least one realization $\mathbf{t} \in \mathbb{C}^n$ where the Jacobian matrix has full row rank.

V. Preliminary Analysis

Lemma 2: For the K-user SISO interference channel with transfer matrix $\mathbf{H}$ of rank D, the sum DoF d_Σ is bounded from above by $\min\{D, K/2\}$, i.e. $d_\Sigma \leq \min\left\{D, \frac{K}{2}\right\}$

Proof: We know that for a generic K-user interference channel the outer bound for the sum DoF is given by $\frac{K}{2}$ [1]. This bound also holds for the rank deficient channel considered in this paper, giving the outer bound of $\frac{K}{2}$ when the rank $D \geq \frac{K}{2}$. The rank of the transfer matrix, D, acts as the cutset bound, *i.e.*, no more than D independent data streams can be transmitted over this network. Hence we get the final outer bound for the sum DoF as, $d_\Sigma \leq \min\{D, \frac{K}{2}\}$. ∎

The outer bound depends on the value of D, but in our analysis we will focus on the setting where $D = \lceil \frac{K}{2} \rceil$. The rank $D = \lceil \frac{K}{2} \rceil$ is the most interesting setting because, if we can prove the achievability for this, the result extends easily to all other values of D. A more rigorous discussion about this can be found in Appendix A.

Consider the determinant of any $l \times l$ sub matrix of the network transfer matrix $\mathbf{H}$, where $l > D$, it gives a polynomial in H_{ij} which identically equates to 0. This implies that the channel coefficients (H_{ij}'s) are algebraically dependent. In a generic interference channel the cross channels are all algebraically independent, so the precoding matrix used in [CJ08] scheme is almost surely full rank. But in a rank deficient interference channel, especially in the case where the rank is $\lceil \frac{K}{2} \rceil$, the cross channels might be algebraically dependent thus making the precoding matrix rank deficient too. We will modify the [CJ08] scheme to exclude the linearly dependent columns in the precoding matrix, which reduces the number of dimensions of the desired and interfering signal spaces at the receivers. We will explore if there is overlap between the desired and the interfering signal spaces, and consequently the achievability of half rate per user.

A. The Modified Scheme

Consider the asymptotic interference alignment scheme for K-user Gaussian SISO interference channel [CJ08] as described in [1], [18]. The symbol extended version of the receiver equation is given as,

$$\mathbf{Y}_j = \sum_{i=1}^{K} \mathbf{H}_{ji}\mathbf{X}_i + \mathbf{Z}_i, \forall j \in \{1, \ldots, K\} \tag{2}$$

Let us denote the precoding matrix used at each transmitter in the [CJ08] scheme as V_n for some arbitrarily large n,

$$V_n = \{(T_1)^{\alpha_1}(T_2)^{\alpha_2} \ldots (T_N)^{\alpha_N}\mathbf{1} \quad | \quad \sum_{i=1}^{N} \alpha_i \leq n, \quad \alpha_1, \alpha_2, \ldots, \alpha_N \in \{0\} \cup Z_+\} \tag{3}$$

$$\mathcal{I}_n = \{(T_1)^{\alpha_1}(T_2)^{\alpha_2} \ldots (T_N)^{\alpha_N}\mathbf{1} \quad | \quad \sum_{i=1}^{N} \alpha_i \leq n+1, \quad \alpha_1, \alpha_2, \ldots, \alpha_N \in \{0\} \cup Z_+\} \tag{4}$$

wherein $T_1, \ldots, T_N$ are the $N = K(K-1)$ cross channels $\mathbf{H}_{ji}, j \neq i$ and $\mathbf{1}$ refers to the all one column vector. We can impose a lexicographic ordering on the columns V_n and $\mathcal{I}_n$. We will construct a new precoding matrix $\bar{V}_n$ by just removing linearly dependent columns of V_n. We will use $\mathcal{I}_n$ and $\bar{\mathcal{I}}_n$ to denote the original interference space and interference space with the modified scheme, at the receivers respectively. Similar to [18], all transmitters use the same set of beamforming vectors $\bar{V}_n$ and all receivers approximately see the same interfering signal space of $\bar{\mathcal{I}}_n$. It can be noted that $\mathcal{I}_n = V_{n+1}$ and $\bar{\mathcal{I}}_n = \bar{V}_{n+1}$. Since we have removed only the linearly dependent columns from V_n and $\mathcal{I}_n$ to form $\bar{V}_n$ and $\bar{\mathcal{I}}_n$, the column span of the precoding matrices remains the same, i.e. following relations hold

$$\begin{aligned} span(\bar{V}_n) &= span(V_n) \\ span(T_i\bar{V}_n) = span(T_iV_n) &\subseteq span(\mathcal{I}_n) = span(\bar{\mathcal{I}}_n) \end{aligned}$$

In the original construction, number of column vectors was given by $|V_n| = \binom{n+N}{N}$ and $|\mathcal{I}_n| = \binom{n+N+1}{N}$. While we do not precisely know the number of column vectors in $\bar{V}_n$ and $\bar{\mathcal{I}}_n$, we know that $|\bar{V}_n| < |\bar{\mathcal{I}}_n| = |\bar{V}_{n+1}|$. Now we will show that the desired signal space occupies half the dimensions at all receivers, almost surely. To this end, we need to align all the interference at every receiver within one half of the total signal space available, leaving the other half interference free for the desired signals. This will enable the receivers to decode its desired message.

We will use limit infimum in proofs for the following lemmas as limits may not exist in general for divergent series.

Lemma 3: Growth rate of the new precoding vectors asymptotically reaches zero for large n, *i.e.*

$$\liminf_{n\to\infty} \frac{|\bar{V}_{n+1}|}{|\bar{V}_n|} = 1 \tag{5}$$

Proof: We will prove this by contradiction. Suppose the contrary is true, i.e., there exists a positive number $\epsilon > 0$ such that

$$\liminf_{n\to\infty} \frac{|\bar{V}_{n+1}|}{|\bar{V}_n|} > (1+\epsilon) \tag{6}$$

By definition of limit infimum, (6) means that there exists a positive integer n_0 such that for all $n > n_0$, the below relation holds.

$$\frac{|\bar{V}_n|}{|\bar{V}_{n_0}|} > (1+\epsilon)^{n-n_0} \tag{7}$$

Note that (7) represents a recursive relation that holds for all positive integers n, leading to :

$$|\bar{V}_n| > (1+\epsilon)^{n-n_0}|\bar{V}_{n_0}| \tag{8}$$

Based on the modified construction of precoding vectors for asymptotic interference alignment scheme, we know that $|\bar{V}_{n+1}| \leq \binom{n+N+1}{N}$. Hence, we have the following :

$$\frac{|\bar{V}_{n+1}|}{|\bar{V}_n|} \leq \frac{\binom{n+N+1}{N}}{(1+\epsilon)^{n-n_0}\bar{V}_{n_0}} \tag{9}$$

It can be seen that for large n, the term on the right side goes to zero since it is a ratio of a polynomial over an exponential in n. Note that we have assumed ϵ to be a positive number. However, we know that this cannot be true since $|\bar{V}_n| \leq |\bar{V}_{n+1}|$, leading to a contradiction. Hence the assumption in (6) cannot hold, and we have proved the lemma, i.e., growth rate of size of precoding matrix after removing the dependent columns, reaches zero asymptotically for large n. ■

Lemma 4: Given that the desired signal space $\bar{V}_n$ does not overlap with the interfering signal space $\bar{\mathcal{I}}_n$, the ratio of desired signal dimensions and total signal dimensions can be made arbitrarily close to $\frac{1}{2}$, *i.e.*

$$\frac{|\bar{V}_n|}{|\bar{V}_n| + |\bar{\mathcal{I}}_n|} \approx \frac{1}{2}$$

Proof: We know from Lemma 3 that (5) holds true. Also, for a sequence x_n, if $a > \liminf x_n$, then there is an infinite subsequence x_{n_k} of x_n such that $a > x_{n_k}$. Using this we can choose a series of n and a value for δ such that

$$1 \leq \frac{|\bar{V}_{n+1}|}{|\bar{V}_n|} < 1 + \delta$$

from which we get

$$\lim_{\delta\to 0} \frac{|\bar{V}_n|}{|\bar{V}_n| + |\bar{V}_{n+1}|} = \frac{1}{2}$$

Hence with appropriate choice of δ, we can make above relation arbitrarily close to $\frac{1}{2}$, *i.e.* the ratio of desired signal dimensions and total signal dimensions reaches $\frac{1}{2}$ for large n. ■

Lemma 3 and 4 imply that for the interference channel with rank deficient transfer matrix, DoF per user can be made arbitrarily close to $\frac{1}{2}$ for large n with the modified scheme, if the desired and interfering signal space do not overlap.

B. The Overlap

Proof of Theorem 1: Consider the signal space at receiver 1, $S_1 = [\mathbf{H}_{11}\bar{V}_n \quad \bar{\mathcal{I}}_n]$. Matrix S_1 needs to be full rank so that desired and interference signal spaces have no overlap.

$$span(\mathbf{H}_{11}\bar{V}_n) \cap span(\bar{\mathcal{I}}_n) = \emptyset \tag{10}$$

Let us denote the number of columns in $\bar{V}_n$ as l_v, and the number of columns in $\bar{\mathcal{I}}_n$ as l_{int} . Note that $l_v = \binom{n+N-1}{N}$ and $l_{int} = \binom{n+N}{N}$ when all cross channels are algebraically independent. Based on modified [CJ08] construction scheme, the linear independence condition can be expressed as

$$\mathbf{H}_{11} \sum_{i=0}^{l_v-1} q_i \prod_m (T_m)^{\alpha_{mi}} \neq \sum_{j=0}^{l_{int}-1} p_j \prod_m (T_m)^{\alpha_{mj}} \tag{11}$$

where $m \in \{1,\ldots,K(K-1)\}$, $\alpha_{mi} \in \{0,1,\ldots,l_v-1\}$, $\alpha_{mj} \in \{0,1,\ldots,l_{int}-1\}$, and all p_i, q_j are not simultaneously zeros. Because of the diagonal nature of $\mathbf{H}_{ij}$'s and T_m's, (11) can be easily translated to its scalar form. If the conditions in Theorem 1 are satisfied, (11) will hold almost surely under $\mathbb{S}$, a rigorous proof for this is presented in Appendix B, and consequently matrix S_1 will be full rank. The same argument can be extended to the other direct channels H_{kk}. This, along with Lemma 4, proves the theorem. ■

Consider the case where the rank is $\lceil \frac{K}{2} \rceil$, *i.e.* $D = \lceil \frac{K}{2} \rceil$. In order to show that the sum DoF outer bound is tight for this case, all we need to prove is that any polynomial of the form $QH_{kk} - P$ is not identically equal to 0 under $\mathbb{S}$. We can make use of the Schwartz-Zippel lemma from polynomial identity testing for this purpose, proof of applicability of Schwartz-Zippel lemma under $\mathbb{S}$ is presented in Appendix B.

Consider the $K \times K$ transfer matrix $\mathbf{H}$, since the rank of this matrix is $D = \lceil \frac{K}{2} \rceil$, it's rank decomposition is the product of a $K \times D$ matrix and a $D \times K$ matrix.

$$\mathbf{H} = \mathbf{G}_{|K\times D|} * \mathbf{F}_{|D\times K|} \tag{12}$$

In one time slot, each receiver will see a linear equation in K variables (messages), each of which is in turn a linear combination of D linear equations. Now consider sending the same set of messages over two consecutive time slots, we will use $\mathbf{H}_1$ to represent the coefficients of the linear equations at the receivers for the first time slot and $\mathbf{H}_2$ for the second time slot. Each receiver would be able to decode its respective message if $\mathbf{H}_1 - \mathbf{H}_2 = \mathbf{I}_{|K\times K|}$, which implies,

$$\begin{aligned} \mathbf{G}_1 * \mathbf{F}_1 - \mathbf{G}_2 * \mathbf{F}_2 &= \mathbf{I}_{|K\times K|} \\ \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_1 \\ -\mathbf{F}_2 \end{bmatrix} &= \mathbf{I}_{|K\times K|} \\ \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{F}_1 \\ -\mathbf{F}_2 \end{bmatrix}^{-1} \end{aligned} \tag{13}$$

wherein $\mathbf{G}_t, \mathbf{F}_t$ are obtained from rank decomposition of matrix $\mathbf{H}_t, t \in \{1,2\}$. If we have the freedom to manipulate $\begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \end{bmatrix}$ or $\begin{bmatrix} \mathbf{F}_1 \\ -\mathbf{F}_2 \end{bmatrix}$, then by choosing one as the inverse of the other and by sending the same message over the two time slots, we would be able to zero force the interference over the two slots at each receiver. This gives us a set of realizations where the value of the cross channels would remain the same while the direct channels would vary, similar to ergodic alignment [15].

We define the subset $\mathbb{S}' \subset \mathbb{S}$ as the set of channel realizations where for each $\mathbf{H}$ in the subset there exists a complementary realization $\mathbf{H}'$ such that $\mathbf{H} - \mathbf{H}' = \mathbf{I}$, i.e.,

$$\mathbb{S}' = \{\mathbf{H} \quad | \quad \mathbf{H} \in \mathbb{S},\ \mathbf{H}' \in \mathbb{S},\ \mathbf{H} - \mathbf{H}' = \mathbf{I}\} \tag{14}$$

When $P \neq 0$ and $Q \neq 0$ under $\mathbb{S}'$, we get non-zero realizations for $QH_{kk} - P$, thus proving that this polynomial is non-zero under $\mathbb{S}$. The same argument could be made when only one of the two polynomials P or Q is non-zero under $\mathbb{S}'$. The problem occurs when both P and Q are zeros in $\mathbb{S}'$, in which case we can not get non-zero realizations for $QH_{kk} - P$ even under $\mathbb{S}'$, making it hard to say whether $QH_{kk} - P$ is a zero or a non-zero polynomial under $\mathbb{S}$. At this point, it is not clear whether the conditions hold for generic K.

VI. Achievability

In this section we will first show the hurdles in proving the sufficient conditions for $K = 4$ and $K = 5$. We will also show that the sufficient conditions are not satisfied for $K \geq 6$.

A. $K = 4$ and $D = 2$

Consider the 4-user rank deficient SISO interference channel with 4 direct channels and 12 cross channels shown below,

$$\mathbf{H} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}$$

Lemma 5: All 12 cross channels of 4×4 channel matrix with rank $D = 2$, are algebraically independent.

Proof: We prove this with the help of symbolic toolbox in MATLAB. Please refer Appendix C1 for further details. ■

Let us denote the first direct channel H_{11} as z; the set of all cross channels as $X = \{H_{ij} : i \neq j; \forall i,j \in \{1,2,3,4\}\}$, and $\mathbb{S}$ the set of all channel realizations with rank 2. Consider a 3×3 submatrix of $\mathbf{H}$ containing two direct channels, say H_{11} (denoted as z) and H_{22}. The determinant of any such submatrix is zero (since $D = 2$).

$$\begin{vmatrix} z & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{41} & H_{42} & H_{43} \end{vmatrix} = 0$$

Evaluating the determinant, we get a polynomial in z, H_{22} and the 7 cross channels. Rearranging the polynomial equation, we can express H_{22} as a rational function, $f_2(z, X)$, of z and the 7 cross channels.

$$H_{22} = \frac{H_{21}(H_{12}H_{43} - H_{42}H_{13}) + H_{23}(zH_{42} - H_{41}H_{12})}{zH_{43} - H_{41}H_{13}}$$

The denominator polynomial, $zH_{43} - H_{41}H_{13}$ is non-zero, this is shown in Appendix C1. The denominator could still evaluate to zero for some realizations in $\mathbb{S}$ and $f_2(z, X)$ will be undefined for these realizations. But we can consider a domain $\mathbb{D}$ under which $f_2(z, X)$ is always defined, *i.e.*, set of all points in $\mathbb{S}$ for which the denominator polynomial is always non-zero. We can also see that the set of points excluded from $\mathbb{S}$ to get $\mathbb{D}$ has measure zero.

Consider the determinant of another 3×3 submatrix comprising of H_{11} and H_{22}.

$$\begin{vmatrix} z & H_{12} & H_{14} \\ H_{21} & f_2(z, X) & H_{24} \\ H_{31} & H_{32} & H_{34} \end{vmatrix} = 0$$

Evaluating the above determinant, we get a multivariate polynomial which is quadratic in z, of the form:

$$A(X)z^2 + B(X)z + C(X) = 0 \tag{15}$$

where $A(X), B(X), C(X)$ are all polynomial functions of the 12 cross channels. Also polynomials $A(X), B(X), C(X)$ are non-zero since the cross channels are algebraically independent.

Let us assume that there is a polynomial $Q(X)H_{11} - P(X)$ that always evaluates to zero under $\mathbb{D}$. We already know that $Q(X)$ is a non-zero polynomial, so we can express H_{11} as a rational function of the cross channels, *i.e.*, $z = H_{11} = \frac{P(X)}{Q(X)}$, which is always defined in the domain $\mathbb{D}' \subseteq \mathbb{D}$. Similar to $\mathbb{D}$, we can see that the set of points excluded from $\mathbb{S}$ to get $\mathbb{D}'$ has measure zero. Substituting this rational function for z in (15), we get

$$A(X)P(X)^2 + B(X)P(X)Q(X) + C(X)Q(X)^2 = 0 \tag{16}$$

The above equation holds if

- The polynomial in (16) is non-trivial and always evaluates to zero.
- $z = \frac{P(X)}{Q(X)}$ is a root of the quadratic equation (15).

If we suppose that the non-trivial polynomial in (16) always evaluates to zero, then this gives a zero polynomial in the 12 cross channels, indicating that the 12 cross channels are algebraically dependent. However this is a contradiction.

If we could show that $z = \frac{P(X)}{Q(X)}$ cannot be a root, we can establish that $Q(X)H_{11} - P(X) \neq 0$. This is hard as we do not exactly know $P(X)$ or $Q(X)$. Using MATLAB, we were able to verify that for rational realizations of the cross channels H_{ij} the roots of the equation (15) are not always rational. Even though this helps in showing that the polynomial $Q(X)H_{11} - P(X)$ is almost surely non-zero in the rational space, we will not be able to make the same statement for the general space $\mathcal{S}$.

B. $K = 5$ and $D = 3$

Consider the 5-user rank deficient SISO interference channel represented by the following matrix,

$$\mathbf{H} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} & H_{15} \\ H_{21} & H_{22} & H_{23} & H_{24} & H_{25} \\ H_{31} & H_{32} & H_{33} & H_{34} & H_{35} \\ H_{41} & H_{42} & H_{43} & H_{44} & H_{45} \\ H_{51} & H_{52} & H_{53} & H_{54} & H_{55} \end{bmatrix}$$

the rank of this matrix is D=3,

Lemma 6: All 20 cross channels along with any one of the direct channel of the 5×5 transfer matrix with rank $D = 3$ are algebraically independent.

Proof: Proof is presented in Appendix C2. ■

Let us denote the first direct channel as $z = H_{11}$, and the fifth direct channel as $Z_{AI} = H_{55}$ (that is algebraically independent of other channels). Set of cross channels is denoted by X, similar to that in $K = 4$ setting.

Consider a 4×4 submatrix of H containing 3 direct channels, say H_{11}, H_{22}, H_{55}, wherein H_{55} is considered to be algebraically independent of all the cross channels. The determinant of any such sub-matrix is zero, since $D = 3$.

$$\begin{vmatrix} z & H_{12} & H_{14} & H_{15} \\ H_{21} & H_{22} & H_{24} & H_{25} \\ H_{31} & H_{32} & H_{34} & H_{35} \\ H_{51} & H_{52} & H_{54} & Z_{AI} \end{vmatrix} = 0$$

Evaluating the determinant, we can express H_{22} as a rational function of z, fifth direct channel Z_{AI} and 20 cross channels.

$$H_{22} = f_2(z, X, Z_{AI}) \tag{17}$$

The denominator polynomial of this rational function can be shown to be non-zero. This is due to the algebraic independence of all cross channels and Z_{AI}, as discussed in Appendix C2. The denominator could still evaluate to zero for some realizations in $\mathbb{S}$ and $f_2(z, X, Z_{AI})$ will be undefined for these realizations. But we can consider a domain $\mathbb{D}$ under which $f_2(z, X, Z_{AI})$ is always defined, *i.e.*, set of all points in $\mathbb{S}$ for which the denominator polynomial is always non-zero.

Now, let us consider determinant of another 4×4 submatrix comprising of H_{11}, H_{22} and H_{55}.

$$\begin{vmatrix} z & H_{12} & H_{13} & H_{15} \\ H_{21} & f_2(z, X, Z_{AI}) & H_{23} & H_{25} \\ H_{41} & H_{42} & H_{43} & H_{45} \\ H_{51} & H_{52} & H_{53} & Z_{AI} \end{vmatrix} = 0$$

Evaluating above determinant, we get a multivariate polynomial which is quadratic in z of the form

$$A(X, Z_{AI})z^2 + B(X, Z_{AI})z + C(X, Z_{AI}) = 0 \tag{18}$$

where $A(X, Z_{AI}), B(X, Z_{AI}), C(X, Z_{AI})$ are all polynomial functions of the 20 cross channels and 1 direct channel H_{55}. Also, $A(X, Z_{AI}), B(X, Z_{AI}), C(X, Z_{AI})$ are non-zero since X, Z_{AI} are algebraically independent.

Let us assume that there is a polynomial $Q(X)H_{11} - P(X)$ that always evaluates to zero under $\mathbb{D}$. We already know that $Q(X)$ is a non-zero polynomial, so we can express H_{11} as a rational function of the cross channels, *i.e.*, $z = H_{11} = \frac{P(X)}{Q(X)}$, which is always defined in the domain $\mathbb{D}' \subseteq \mathbb{D}$. Similar to $\mathbb{D}$, we can see that the set of points excluded from $\mathbb{S}$ to get $\mathbb{D}'$ has measure zero. Substituting this rational function for z in (18), we get

$$A(X, Z_{AI})\frac{p(X)^2}{q(X)^2} + B(X, Z_{AI})\frac{p(X)}{q(X)} + C(X, Z_{AI}) = 0 \quad (19)$$

Above equation holds if

- The polynomial in (19) is non-trivial and always evaluates to zero.
- $z = \frac{P(X)}{Q(X)}$ is a root of the quadratic equation (18).

If the non-trivial polynomial in (19) always evaluates to zero, then it gives a zero polynomial in the 20 cross channels and 1 direct channel, indicating that the 21 channels are algebraically dependent. However this is a contradiction.

If we could show that $z = \frac{P(X)}{Q(X)}$ cannot be a root, we can establish that $Q(X)H_{11} - P(X) \neq 0$. This is hard as we do not exactly know $P(X)$ or $Q(X)$. Using MATLAB, we could verify that for rational realizations of the 20 cross channels H_{ij} and 1 direct channel, the roots of the equation (18) are not always rational. Even though this helps in showing that the polynomial $Q(X)H_{11} - P(X)$ is almost surely non-zero in the rational space, we will not be able to make the same statement for the general space $\mathcal{S}$.

This analysis provides insights into why it is hard to prove the sufficient conditions, even for the simple 4-user and 5-user channels.

C. Complications with Higher Number of Users

The 4-user and 5-user channels have algebraically independent cross channels. But as we increase the number of users to 6, we can see that the cross channels are no longer algebraically independent. To see how this might affect us, consider the 6-user interference channel with rank deficient transfer matrix $\mathbf{H}$. Similar to the analysis in section VI-A, consider a 4×4 submatrix of $\mathbf{H}$ containing only 2 direct channels, say $H_{11} = z$, and H_{22}, the determinant of any such sub matrix is zero. Evaluating this determinant, we can express H_{22} as a rational function of H_{11} and the 12 cross channels, $H_{22} = f_2(z, X)$, and this rational function is always defined in a domain $\mathbb{D}$.

Consider the determinant of another 4×4 submatrix containing of H_{11} and $H_{22} = f_2(z, X)$. Evaluating this determinant, we can get a multivariate polynomial which is quadratic in z of the form

$$A_1(X)z^2 + B_1(X)z + C_1(X) = 0 \quad (20)$$

where $A(X), B(X), C(X)$ are all polynomial functions of the cross channels. We can do the same for H_{33}, by considering two different sub matrices containing H_{11} and H_{33}, and derive another multivariate polynomial which is quadratic in z after modifying the domain $\mathbb{D}$ to include the rational function $H_{33} = f_3(z, X)$.

$$A_2(X)z^2 + B_2(X)z + C_2(X) = 0 \quad (21)$$

In the 4-user case, doing this with H_{33} would result in the same polynomial as (15). But in the 6-user case considered here, we can see that (20) will have certain cross channel coefficients, namely H_{24} and H_{42}, which are not present in (21), and (21) will have certain cross channel coefficients, namely H_{34} and H_{43}, which are not present in (20). By linearly combining (20) and (21) after scaling them appropriately, we can eliminate the z^2 terms and solve for z as a function of the cross channels.

$$(A_2(X)B_1(X) - A_1(X)B_2(X))z - (A_1(X)C_2(X) - A_2(X)C_1(X)) = 0 \quad (22)$$

The above equation shows that for a 6-user interference channel with rank $D = 3$, there exists a polynomial $Q(X)H_{11} - P(X) = 0$. But, both the $P(X)$ part and $Q(X)$ part of (22) have to be non-zero polynomials in order for (22) to be relevant, as under the modified alignment scheme it is not possible for either of them to be zero polynomials. That being said, by using MATLAB we can numerically confirm that these polynomials are non-zeros and thus (22) is relevant. In other words the desired and interfering signal spaces will overlap at the receivers if we try to use the modified scheme for the 6-user channel.

Even though this is not enough to say that the DoF outer bounds from Lemma 2 are not achievable for higher number of users, it shows the complications that arise when we increase the number of users in the interference channel with rank deficient transfer matrix.

VII. Conclusions

We introduced the problem of characterizing the DoF of the K-user interference channel with rank a deficient transfer matrix. We presented a modified asymptotic alignment scheme to handle the algebraic dependencies, and discussed the sufficient conditions to achieve half rate per user. We illustrated the difficulties of proving the sufficient conditions for the simpler cases of $K = 4, 5$ and showed that the sufficient conditions are not met for $K \geq 6$. In conclusion, finding the optimal DoF of the general K-user interference channel with rank deficient transfer matrix remains open and presents a considerable challenge.

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APPENDIX

A. Achievability for other values of D

In this section we will show how to prove the *DoF* achievability when the rank of the transfer matrix is either less than $\lceil \frac{K}{2} \rceil$ or greater than $\lceil \frac{K}{2} \rceil$, *i.e.* $D < \lceil \frac{K}{2} \rceil$ or $D > \lceil \frac{K}{2} \rceil$, provided we could show the achievability for $D = \lceil \frac{K}{2} \rceil$. We will assume that DoF of $1/2$ per user is achievable when $D = \lceil \frac{K}{2} \rceil$ throughout this section.

First let us consider $D < \lceil \frac{K}{2} \rceil$ case, the DoF outer bound here is simply D. The achievability for this case is based on the achievable scheme for a $K = 2D$ user interference channel with rank D transfer matrix. Consider the K user channel, if we choose only $2D$ users among the K users to be active at any given time, it reduces to the case of $D = K/2$ thus making it possible to achieve the sum *DoF* of D. Symmetric DoF can be achieved by cycling through the $\binom{K}{2D}$ combinations of active users. Here each user is active in $\binom{K-1}{2D-1}$ combinations and a *DoF* of half per user is assumed to be achievable in combination, thus each user will get total *DoF* of $\frac{1}{2} \times \frac{\binom{K-1}{2D-1}}{\binom{K}{2D}} = \frac{D}{K}$.

Now consider the $D > \lceil \frac{K}{2} \rceil$ case, we will show that a sum *DoF* of $K/2$ is achievable using the scheme from section V-A provided it works for $D = \lceil \frac{K}{2} \rceil$. More specifically, we will assume that the sufficient conditions in Theorem 1 hold true. We will make use of concepts of Varieties and Ideals from algebraic geometry to prove this case. We know that the determinant of any $(D+1) \times (D+1)$ sub matrix is going to be zero. Let $\mathbb{V}_D$ denote the *affine variety* generated by these determinant polynomials, note that the $\mathbb{S}$ defined in section III is also an variety and $\mathbb{S} = \mathbb{V}_D$. Without loss of generality consider the case of $D = \lceil \frac{K}{2} \rceil + 1$, let $\mathbb{V}_{\lceil \frac{K}{2} \rceil + 1}$ denote the *affine variety* and $I_{\lceil \frac{K}{2} \rceil + 1}$ can be used to represent the ideal generated by this variety, *i.e.* $I_{\lceil \frac{K}{2} \rceil + 1} = I(\mathbb{V}_{\lceil \frac{K}{2} \rceil + 1})$. The following are true,

$$\mathbb{V}_{\lceil \frac{K}{2} \rceil} \subseteq \mathbb{V}_{\lceil \frac{K}{2} \rceil + 1} \tag{23}$$

The above equation implies that,

$$I_{\lceil \frac{K}{2} \rceil} \supseteq I_{\lceil \frac{K}{2} \rceil + 1} \tag{24}$$

Now assume, under $D = \lceil \frac{K}{2} \rceil + 1$, the direct channels can be expressed as a rational function of the cross channels. This tells us that there exists a polynomial, let us call it f_1, of the form $q(X)H_{kk} - p(X)$ that evaluates to 0, *i.e.* $f_1 = 0$, for any realization in the variety, where X is the vector of cross channels and $p(X)$ and $q(X)$ are multivariate polynomials in the cross channels. Since $f_1 = 0$ under the affine variety, we get $f_1 \in I_{\lceil \frac{K}{2} \rceil + 1}$, which implies $f_1 \in I_{\lceil \frac{K}{2} \rceil}$. This contradict our primary assumption that direct channels cannot be expressed as a rational function of the cross channels under $D = \lceil \frac{K}{2} \rceil$, thus we can conclude that direct channels cannot be expressed as a rational function of the cross channels even under $D = \lceil \frac{K}{2} \rceil + 1$. This argument can be extended to all cases of $D > \lceil \frac{K}{2} \rceil$.

B. Schwartz-Zippel Lemma for the Variety

We have a transfer matrix $\mathbf{H} = [H_{ij}]$ of size $K \times K$ and rank D. The sample space, $\mathbb{S}$ here is the set of all channel realizations for which the rank of the transfer matrix does not exceed D, and as seen in Appendix A this is same as $\mathbb{V}_D$. Since the sample space $\mathbb{S}$ is an affine variety and not a field, it is not clear how Schwartz-Zippel lemma would be applicable here. In this section we will show that the Schwartz-Zippel lemma is valid even under $\mathbb{S}$.

The transfer matrix $\mathbf{H}$ can be written as the product of a $K \times D$ matrix $\mathbf{G}$ and a $D \times K$ matrix $\mathbf{F}$.

$$\begin{aligned}\mathbf{H} &= \mathbf{G}_{|K\times D|} * \mathbf{F}_{|D\times K|} \\ &= \begin{bmatrix} g_{11} & \cdots & g_{1D} \\ g_{21} & \ddots & g_{2D} \\ \vdots & \ddots & \vdots \\ g_{K1} & \cdots & g_{KD} \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1K} \\ \vdots & \ddots & \ddots & \vdots \\ f_{D1} & f_{D2} & \cdots & f_{DK} \end{bmatrix}\end{aligned}$$

Each of the channel coefficient (H_{ij}) in $\mathbf{H}$ can be expressed as a polynomial of certain g_{ij}'s and f_{ij}'s, namely $H_{ij} = \sum_{k=1}^{D} g_{ik} f_{kj}$, this is a parametric representation of H_{ij}. This implies that all realizations of $\mathbf{H}$ are given by $\mathbf{G} * \mathbf{F}$, as g_{ij}'s and f_{ij}'s vary over $\mathbb{C}$.

Consider a non-trivial polynomial in H_{ij}'s, that evaluates to zero always. Substituting H_{ij} with its parametric form, we can see that the polynomial will trivially reduce to zero. But for a non-zero polynomial in H_{ij}'s, we can see that the parametric representation of the polynomial will not trivially reduce to zero, i.e., we will have a non-zero polynomial in f_{ij}'s and g_{ij}'s instead. Since the f_{ij}'s and g_{ij}'s take values from the filed $\mathbb{C}$, we can now use Schwartz-Zippel lemma for any polynomials in these variables.

C. Algebraic Independence of Channels

For a K user interference channel with a transfer matrix of rank $D = \lceil K/2 \rceil$, the transfer matrix $\mathbf{H}$ can be written as the product of a $K \times D$ matrix $\mathbf{G}$ and a $D \times K$ matrix $\mathbf{F}$.

$$\begin{aligned} \mathbf{H} &= \mathbf{G}_{|K\times D|} * \mathbf{F}_{|D\times K|} \\ &= \begin{bmatrix} g_{11} & \cdots & g_{1D} \\ g_{21} & \ddots & g_{2D} \\ \vdots & \ddots & \vdots \\ g_{K1} & \cdots & g_{KD} \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1K} \\ \vdots & \ddots & \ddots & \vdots \\ f_{D1} & f_{D2} & \cdots & f_{DK} \end{bmatrix} \end{aligned}$$

The elements of $\mathbf{G}$ and $\mathbf{F}$ form an algebraically independent set since they are generic variables. We have $K(K-1)$ cross channel coefficients each of which can be expressed as a polynomial function of the generic variables g_{ij}'s and f_{ij}'s.

1) $K = 4$ *and* $D = 2$ *:*

Proof of lemma 5: The transfer matrix for the 4 user case has 12 cross channels and there are 16 generic variables as the rank $D = 2$. In order to prove the 12 cross channels are algebraically independent, we write down the 12×16 Jacobian matrix as described in lemma 1. We need to prove the rank of this matrix is almost always 12 and for that all we need to show is a single realization of g_{ij}'s and f_{ij}'s that gives a non zero value for the determinant of any 12×12 sub matrix. The idea here is that the determinant of the jacobian matrix is a multivariate polynomial in g_{ij}'s and f_{ij}'s and using the Schwartz-Zippel lemma we can argue that if this polynomial is has a non-zero realization that almost surely the polynomial by itself is non-zero. Using MATLAB, we can see that the determinant polynomial of the Jacobian matrix is non-zero for a random realization of g_{ij}'s and f_{ij}'s. ■

Similar to the proof of lemma 5, we can also show that the channels H_{11}, H_{13}, H_{41} and H_{43} are algebraically independent. This in turn shows that the polynomial $zH_{43} - H_{41}H_{13}$ is non-zero.

2) $K = 5$ *and* $D = 3$ *:*

Proof of lemma 6: The transfer matrix for this case has 20 cross channels and the number of generic variables here is 30. Here we consider 21 channels of the transfer matrix comprising of 20 cross channels and anyone of the direct channels, say H_{55}, without loss of generality. The proof for this is similar to the case in C1, we just consider 1 direct channel along with 20 cross channels. The matrix from lemma 1 is of size 21×30 for this case. Similar to the 4-user case, using MATLAB we can see that the determinant polynomial of the Jacobian matrix is non-zero for a random realization of g_{ij}'s and f_{ij}'s. ■

Similar to the proof of lemma 6, we can also show that the channels involved in the denominator polynomial of $f_2(z, X, Z_{AI})$ in equation 17, are algebraically independent. This in turn shows that the denominator polynomial of $f_2(z, X, Z_{AI})$ is non-zero.