Abstract—In this paper, we consider the problem of multiple unicast sessions over a directed acyclic graph. It is well known that linear network coding is insufficient for achieving the whole rate region, in the general case. However, there exist networks for which routing is sufficient to achieve the whole rate region, and we refer to them as routing-optimal networks. We identify a class of routing-optimal networks, which we refer to as information-distributive networks, defined by three topological features. Due to these features, for each rate vector achieved by network coding, there is always a routing scheme such that it achieves the same rate vector, and the traffic transmitted through the network is exactly the information transmitted over the cut-sets between the sources and the sinks in the corresponding network coding scheme. We present examples of information-distributive networks, including some examples from (1) index coding and (2) from a single unicast session with hard deadline constraint.

I. INTRODUCTION

In this paper, we consider network coding for multiple unicast sessions over directed acyclic graphs. In general, non-linear network coding should be considered in order to achieve the whole rate region of network coding [1]. Yet, there exist networks, for which routing is sufficient to achieve the whole rate region. We refer to these networks as routing-optimal networks. We attempt to answer the following questions: 1) What are the distinct topological features of these networks? 2) Why do these features make a network routing-optimal? The answers to these questions will not only explain which kind of networks can or cannot benefit from network coding, but will also deepen our understanding on how network topologies affect the rate region of network coding.

A major challenge is that there is currently no effective method to calculate the rate region of network coding. Some researchers proposed to use information inequalities to approximate the rate region [2]. However, except for very simple networks, it is very difficult to use this approach since there is potentially an exponential number of inequalities that need to be considered. [3] provides a formula to calculate the rate region by finding all possible entropy functions, which are vectors of an exponential number of dimensions, thus very difficult to solve even for simple networks.

In this paper, we employ a graph theoretical approach in conjunction with information inequalities to identify topological features of routing-optimal networks. Our high-level idea is as follows. Consider a network code. For each unicast session, we choose a cut-set \( C \) between source and sink, and a set \( P \) of paths from source to sink such that each path in \( P \) passes through an edge in \( C \). Since the information transmitted from the source is totally contained in the information transmitted along the edges in \( C \), we can think of distributing the source information along the edges in \( C \) (details will be explained later). Moreover, we consider a routing scheme in which the traffic transmitted along each path \( P \in P \) is exactly the source information distributed over the edge in \( C \) that is traversed by \( P \). Such a routing scheme achieves the same rate vector as the network code. However, since the edges might be shared among multiple unicast sessions, such a routing scheme might not satisfy the edge capacity constraints. This suggests that the cut-sets and path-sets we choose for the unicast sessions should have special features. These are essentially the features we are looking for to describe routing-optimal networks.

We make the following contributions:

- We identify a class of networks, called information-distributive networks, which are defined by three topological features. The first two features capture how the edges in the cut-sets are connected to the sources and the sinks, and the third feature captures how the paths in the path-sets overlap with each other. Due to these features, given a network code, there is always a routing scheme such that it achieves the same rate vector as the network code, and the traffic transmitted through the network is exactly the source information distributed over the cut-sets between the sources and the sinks.
- We prove that if a network is information-distributive, it is routing-optimal. We also show that the converse is not true. This indicates that the three features might be too restrictive in describing routing-optimal networks.
- We present examples of information-distributive networks taken from the index coding problem [4] and single unicast with hard deadline constraint.

We expect that our work will provide helpful insights towards characterizing all possible routing-optimal networks.

II. PRELIMINARIES

A. Network Model

The network is represented by an acyclic directed multi-graph \( G = (V,E) \), where \( V \) and \( E \) are the set of nodes and the set of edges in the network respectively. Edges are denoted by \( e = (u,v) \in V \times V \times Z_{\geq 0} \), or simply by \((u,v)\), where \( v = \text{head}(e) \) and \( u = \text{tail}(e) \). Each edge represents an error-free and delay-free channel with capacity rate of one. Let \( \text{In}(v) \) and \( \text{Out}(v) \) denote the set of incoming edges and the set of outgoing edges at node \( v \) respectively.

There are \( K \geq 1 \) unicast sessions in the network. The \( i \)-th unicast session is denoted by a tuple \( \omega_i = (s_i,d_i) \), where \( s_i \) and \( d_i \) are the source and the sink of \( \omega_i \) respectively. The message sent from \( s_i \) to \( d_i \) is assumed to be a uniformly distributed random variable \( Y_i \) with finite alphabet \( Y_i = \{1,\cdots,\lceil 2^{nR_i} \rceil \} \), where \( R_i \) is the source information rate at \( s_i \). All \( Y_i \)'s are mutually independent. Given \( 1 \leq i \leq j \leq K \), denote \( Y_{ij} = \{ Y_m : i \leq m \leq j \} \). We assume \( \text{In}(s_i) = \text{Out}(d_i) = \emptyset \) for all \( 1 \leq i \leq K \).

Let \( \text{mincut}(u,v,G) \) denote the minimum capacity of all cut-sets between two nodes \( u \) and \( v \). Given two nodes \( u,v \), let \( P_{uv} \) denote the set of directed paths from \( u \) to \( v \). The routing domain of \( \omega_i \), denoted by \( G_i \), is the sub-graph induced by the edges of the paths in \( P_{s_id_i} \).

B. Routing Scheme

A routing scheme is a transmission scheme where each node only replicates and forwards the received messages onto its
outgoing edges. Define the following linear constraints:
\[ \sum_{i=1}^{K} f_i(P) \leq 1 \quad \forall e \in E \]  
where \( f_i(P) \in \mathbb{R}_{\geq 0} \) represents the amount of traffic routed through path \( P \) for \( e \). A rate vector \( \mathbf{R} = (R_i : 1 \leq i \leq K) \in \mathbb{R}_{K} \) is achievable by routing scheme if there exist \( f_i(P) \)'s such that (1) and (2) are satisfied. The rate region of routing scheme, denoted by \( \mathcal{R}_r \), is the set of all rate vectors achievable by routing scheme.

C. Network Coding Scheme

A network coding scheme is defined as follows: [3]

**Definition 1.** An \((n_i, \eta_i : e \in E), (R_i : 1 \leq i \leq K), (\delta_i : 1 \leq i \leq K)) \) network code with block length \( n \) is defined by:

1. for each \( 1 \leq i \leq K \) and \( e \in \text{Out}(s_i) \), a local encoding function: \( \phi_i : \mathcal{Y}_i \to \{1, \ldots, \eta_i\} \);
2. for each \( v \in V - \{s_i, d_i : 1 \leq i \leq K\} \) and \( e \in \text{Out}(v) \), a local encoding function: \( \phi_i : \prod_{e \in \text{In}(v)} \{1, \ldots, \eta_i\} \to \{1, \ldots, \eta_i\} \);
3. for each \( 1 \leq i \leq K \), a decoding function: \( \psi_i : \prod_{e \in \text{In}(d_i)} \{1, \ldots, \eta_i\} \to \mathcal{Y}_i \);
4. for each \( 1 \leq i \leq K \), the decoding error for \( \omega_i \) is \( \delta_i = \Pr[\hat{\psi}_i(Y_1:Y_i) \neq Y_i] \), where \( \psi_i(Y_1:Y_i) \) is the value of \( \psi_i \) as a function of \( Y_1:Y_i \).

Given \( e \in E \), let \( \bar{\omega}_e = \bar{\phi}_e(Y_1:Y_i) \), where \( \bar{\phi}_e(Y_1:Y_i) \) is the value of \( \phi_e \) as a function of \( Y_1:Y_i \), denote the random variable transmitted along edge \( e \) in a network code. For a subset \( C \subseteq E \), denote \( \mathcal{U}_C = \{u_e : e \in C\} \).

**Definition 2.** A rate vector \( \mathbf{R} = (R_i : 1 \leq i \leq K) \in \mathbb{R}^K \) is achievable by network coding if for any \( \epsilon > 0 \), there exists for sufficiently large \( n \), an \((n_i, \eta_i : e \in E), (R_i : 1 \leq i \leq K), (\delta_i : 1 \leq i \leq K)) \) network code such that the following conditions are satisfied:

\[ \frac{1}{n} \log \eta_e \leq 1 + \epsilon \quad \forall e \in E \]  
\[ R_i \geq R'_i - \epsilon \quad \forall 1 \leq i \leq K \]  
\[ \delta_i \leq \epsilon \quad \forall 1 \leq i \leq K \]  

The capacity region achieved by network coding, denoted by \( \mathcal{R}_{nc} \), is the set of all rate vectors \( \mathbf{R} \) achievable by network coding.

Given a network code that satisfies (3)-(5), the following inequalities must hold:

\[ \frac{1}{n} H(U_e) \leq \frac{1}{n} \log(\eta_e) \leq 1 + \epsilon \quad \forall e \in E \]  
\[ \frac{1}{n} H(Y_i) = \frac{1}{n} \log([\eta_i^{R_i}]) \geq R_i \geq R'_i - \epsilon \quad \forall 1 \leq i \leq K \]  
\[ \frac{1}{n} I(Y_i; U_{\text{In}(d_i)}) \geq (1 - \epsilon)(R'_i - \epsilon) - \frac{1}{n} \quad \forall 1 \leq i \leq K \]  

where (8) is due to Fano’s Inequality.

\[ \frac{1}{n} I(Y_1; U_{\text{In}(d_i)}) \geq \frac{1}{n} (H(Y_1) - \delta_i \log |Y_i| - 1) \]

\[ = \frac{1}{n} (1 - \delta_i)H(Y_1) - \frac{1}{n} \geq (1 - \epsilon)(R'_1 - \epsilon) - \frac{1}{n} \]

D. Routing-Optimal Networks

Since routing scheme is a special case of network coding, \( \mathcal{R}_r \subseteq \mathcal{R}_{nc} \).

**Definition 3.** A network is said to be routing-optimal, if \( \mathcal{R}_{nc} = \mathcal{R}_r \), i.e., for such network, routing is sufficient to achieve the whole rate region of network coding.

III. A CLASS OF ROUTING-OPTIMAL NETWORKS

In this section, we present a class of routing-optimal networks, called information-distributive networks. We defer proofs to [5].

A. Illustrative Examples

**Example 1.** We start with the simplest case of single unicast. It is well known that for this case, a network is always routing-optimal [6]. In this example, we re-investigate this case from a new perspective in order to highlight some of the important features that make it routing optimal. Let \( m = \min\{s_1, d_1, G\} \), and \( C = \{e_1, \ldots, e_m\} \) is a cutset between \( s_1 \) and \( d_1 \). Assume \( R'_1 \in \mathcal{R}_{nc} \). Therefore, for every \( \epsilon > 0 \), there exists a network code such that (3)-(5) are satisfied. In the following, all the random variables are defined in this network code.

One important feature of this network is that each path from \( s_1 \) to \( d_1 \) must pass through at least an edge in \( C \). Thus, \( U_{\text{In}(d_1)} \) is a function of \( U_C \). The following inequality holds:

\[ I(Y_1; U_{\text{In}(d_1)}) \leq I(Y_1; U_C) \]  

The following equation holds:

\[ I(Y_1; U_C) = \sum_{j=1}^{m} I(Y_1; U_{e_j}|U_{(e_1,\ldots,e_{j-1})}) \]  

Intuitively, we can interpret (10) as follows: \( I(Y_1; U_{e_j}) \) is the amount of information about \( Y_1 \) that can be obtained from \( U_{e_j} \), \( I(Y_1; U_{e_j}|U_{e_{j+1}}|\ldots|U_{e_m}) \) the amount of information about \( Y_1 \) that can be obtained from \( U_{e_j} \), excluding those already obtained from \( U_{e_{j+1}} \), \ldots, \( U_{e_m} \), and so on. Hence, (10) can be seen as a “distribution” of the source information over the edges in \( C \). Moreover, for each \( 1 \leq j \leq m \), we have:

\[ I(Y_1; U_{e_j}|U_{(e_1,\ldots,e_{j-1})}) \leq H(U_{e_j}) \]  

Another important feature is that due to Menger’s Theorem, there exist \( m \) edge-disjoint paths, \( P_1, \ldots, P_m \), from \( s_1 \) to \( d_1 \) such that \( e_j \in P_j \) for \( 1 \leq j \leq m \). Due to this feature, we can construct a routing scheme by simply letting each \( P_j \) transmit the information distributed on \( e_j \):

\[ f^{n,k}(P) = \begin{cases} \frac{1}{n} I(Y_1; U_{e_j}|U_{(e_1,\ldots,e_{j-1})}) & \text{if } P = P_j, 1 \leq j \leq m \\ 0 & \text{otherwise} \end{cases} \]  

Clearly, due to (6) and (11), the above routing scheme satisfies the following inequalities:

\[ f^{n,k}(P_j) \leq \frac{1}{n} H(U_{e_j}) \leq 1 + \frac{1}{k} \]  

Moreover, due to (8)-(10), we have:

\[ \sum_{j=1}^{m} f^{n,k}(P) = \sum_{j=1}^{m} f^{n,k}(P_j) = \frac{1}{n} I(Y_1; U_C) \geq \left(1 - \frac{1}{k} \right) \left( R'_1 - \frac{1}{k} \right) - \frac{1}{n} \]  

Since \( f^{n,k}(P) \) have an upper bound (see (13)), there exists a sub-sequence \( (n_i, k_i)_{i=1}^{\infty} \) such that each sequence \( (f^{n_i,k_i}(P))_{i=1}^{\infty} \) approaches a finite limit. Define the following routing scheme:

\[ f_i(P) = \begin{cases} \lim_{i \to \infty} f^{n_i,k_i}(P) & \text{if } P = P_j(1 \leq j \leq m) \\ 0 & \text{otherwise} \end{cases} \]  

Due to (13) and (14), the above routing scheme satisfies (1) and (2). Hence, \( R'_1 \in \mathcal{R}_r \), which implies \( \mathcal{R}_{nc} \subseteq \mathcal{R}_r \). Therefore, the network is routing-optimal.

As shown above, two features are essential in making a network with single-unicast routing-optimal. The first feature is the existence of a cut-set such that each path from the source
to the sink must pass through an edge in the cut-set. Due to this feature, the source information contained in $U_{ini(d_i)}$ can be completely obtained from the messages transmitted through the cut-set $C$ (see (9)). The second feature is the existence of edge-disjoint paths $P_1, \cdots, P_m$, each of which passes through exactly one edge in $C$. Due to this feature, a routing scheme can be constructed such that the traffic transmitted along the paths $P_1, \cdots, P_m$ is exactly the information distributed on the edges in $C$ (see (12)). These two features together guarantee that the routing scheme achieves the same rate as network coding (see (13), (14)).

However, extending these features to multiple unicast sessions is not straightforward. One difference from single unicast is that $U_{ini(d_i)}$ may not be a function of $U_C$, where $C$ is a cut-set between $s_i$ and $d_i$, and thus (9) might not hold. Another difference is that the information from multiple unicast sessions might be distributed on an edge, and thus (11) might not hold. Moreover, the paths for multiple unicast sessions might overlap with each other, and thus (13) might not hold. These differences suggest that the cut-sets and the paths, over which a routing scheme is to be constructed, should have additional features in order for the resulting routing scheme to achieve the same rate vector as network coding. We use an example to illustrate some of these features.

**Example 2.** Consider the network shown in Fig. 1a. Consider an arbitrary rate vector $R = (R_1^1, R_2^2) \in R_{nc}$. Therefore, for $\epsilon = \frac{1}{3}$ ($k \in \mathbb{Z}_{>0}$), there exists a network code that satisfies (3)-(5). In the sequel, all the random variables are defined in Fig. 1. Examples of information-distributive networks, where $s_i, d_i$ (for $1 \leq i \leq 3$) are the source and the sink of the $i$th unicast session respectively.

Moreover, $Out(s_1) \cup C_2$ is a cut-set between $s_1, s_2$ and $d_2$, and $U_{Out(s_1)}$ is a function of $Y_1$. Hence $U_{ini(d_i)}$ is a function of $Y_1, U_C$, which implies:

$$I(Y_1; U_{ini(d_i)}) \leq I(Y_1; U_C)$$

We choose the cut-set $C = \{e_1, e_2, e_3\}$ between $s_1$ and $d_1$, and a set of paths $P_1 = \{P_{11}, P_{12}, P_{13}\}$ that pass through $e_1, e_2, e_3$ respectively; for $\omega_2$, we choose a cut-set $C_2 = \{e_2, e_3\}$ between $s_2$ and $d_2$, and a set of paths $P_2 = \{P_{21}, P_{22}\}$ that pass through $e_2, e_3$ respectively.

We first investigate $C_1, C_2$. One important feature is that each path from $s_2$ to $d_1$ passes through at least an edge in $C_1$. Thus, $C_1$ is also a cut-set between $s_1, s_2$ and $d_1$, and $U_{ini(d_i)}$ becomes a function of $UC_1$. Hence, we have:

$$I(Y_1; U_{ini(d_i)}) \leq I(Y_1; U_C)$$

Moreover, $Out(s_1) \cup C_2$ is a cut-set between $s_1, s_2$ and $d_2$, and $U_{Out(s_1)}$ is a function of $Y_1$. Hence $U_{ini(d_i)}$ is a function of $Y_1, U_C$, which implies:

$$I(Y_2; U_{ini(d_i)}) \leq I(Y_2; U_C)$$

We distribute the source information over $C_1, C_2$ as follows:

$$I(Y_1; U_{C_1}) = I(Y_1; U_{e_2}) + I(Y_1; U_{e_1})$$

Another feature about $C_1, C_2$ is that edge $e_1$ is connected to only one source $s_1$, and thus $U_{e_1}$ is a function of $Y_1$. As shown below, this feature guarantees that the information distributed on an edge $e \in C_1 \cup C_2$ is completely contained in $U_e$. First, for $e_1$, it can be easily seen that:

$$I(Y_1; U_{e_1}) \leq H(U_{e_1})$$

For $e_2$, we have:

$$I(Y_1; U_{e_2}|U_{e_1}) + I(Y_2; U_{e_2}|Y_1)$$

where ($b$) is due to the fact that $U_{e_2}$ is a function of $Y_1$, and thus, $I(Y_2; U_{e_2}|Y_1) = I(Y_2; U_{e_2}|Y_1, U_{e_1})$. Similarly, for $e_3$, we have:

$$I(Y_1; U_{e_3}|U_{e_1}, e_2) + I(Y_2; U_{e_3}|Y_1, U_{e_1}, e_2)$$

where ($c$) is again due to the fact that $U_{e_1}$ is a function of $Y_1$.

Next, we investigate $P_1, P_2$. One important feature is that if $P \in P_1$ overlaps with $P' \in P_2$, $P \cap C = P' \cap C$. For example, $P_2$ overlaps with $P_{11}$, and $P_{12}$ contains $P_2 = \{e_2\}$. This feature ensures that the information distributed over $C_1, C_2$ can be further distributed over the paths in $P_1, P_2$. To see this, we construct the following routing scheme:

$$f_{1, k}^P(p) = \begin{cases} \frac{1}{n} I(Y_1; U_{e_1}|U_{e_1, \cdots, e_{j-1}}) & \text{if } P = P_{1j}, 1 \leq j \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

For $e_4$, we have:

$$f_{1, k}^P(p) = \begin{cases} \frac{1}{n} I(Y_1; U_{e_1}) & \text{if } P = P_{21} \\ \frac{1}{n} I(Y_1; U_{e_3}) & \text{if } P = P_{22} \\ 0 & \text{otherwise.} \end{cases}$$

Due to (18)-(20), we can derive that for each $e \in C_1 \cup C_2$, $f_{1, k}^P(p)$ approximately approaches a finite limit. Define a routing scheme:

$$f_i(p) = \begin{cases} \lim_{n \to \infty} f_{i, k}^P(p) & \text{if } P \in P_i, i = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Due to (21) and (22), $f_i(p)$ satisfies (1) and (2). Hence, $R \in R_r$, and $R_{nc} \subseteq R_r$. The network is routing-optimal.

**B. Information Distributive Networks**

In this subsection, we present the definition of information-distributive networks. Similarly to single unicast, for each unicast session $\omega_i$ ($1 \leq i \leq K$), we choose a cut-set $C_i$ between $s_i$ and $d_i$, and a set of paths $P_i$ from $s_i$ to $d_i$. The collection of these cut-sets, denoted by $W = (C_i)_{i=1}^K$, is called a cut-set sequence, and the collection of these path-sets, denoted by $K = (P_i)_{i=1}^K$, is called a path-sequence. For instance, in Example 2, we choose a cut-set sequence $W = (C_i)_{i=1}^K$, where $C_i = \{e_1, e_2, e_3\}$ is a cut-set between $s_i$ and $d_i$. 
and $C_2 = \{e_2, e_3\}$ is a cut-set between $s_2$ and $d_2$, and a path-set sequence $K = \{P_i\}_{i=1}^{21}$, where $P_1$ is a path-set from $s_1$ to $d_1$, and $P_2$ a path-set from $s_2$ to $d_2$. Moreover, we arrange the edges in each cut-set in $W$ in some ordering. For instance, in Example 2, we arrange the edges in $C_1$ in the ordering $T_1 = (e_1, e_2, e_3)$, and the edges in $C_2$ in the ordering $T_2 = (e_2, e_3)$. Each such ordering is called a permutation of the edges in the corresponding cut-set. The collection of these permutations, denoted $\mathcal{T} = \{T_i\}_{i=1}^{21}$, is called a permutation sequence. For $e \in C_i$, let $T_i(e)$ denote the subset of edges before $e$ in $T_i$. For $e \in E$, define $W(e) = \{C_i \in \mathcal{W} : e \in C_i\}$, and $\alpha(e)$ the largest index of the source to which tail(e) is connected. The first feature is described below.

**Definition 4.** Given a cut-set sequence $\mathcal{W}$, if for all $1 \leq i < j \leq K$, each path from $s_i$ to $d_i$ must pass through an edge in $C_j$, we say that $\mathcal{W}$ is cumulative.

This feature guarantees that the source information contained in the incoming messages at each sink $d_i$ can be completely obtained from $Y_{1:i-1}, U_{C_i}$.

**Lemma 1.** Consider a network code as defined in Definition 1. If $\mathcal{W}$ is a cumulative cut-set sequence, then for each $1 \leq i \leq K$, $Y_i$ is a function of $Y_{1:i-1}, U_{C_i}$, and we get:

$$I(Y_i; U_{\text{in}(d_i)}|Y_{1:i-1}) \leq I(Y_i; U_{C_i}|Y_{1:i-1})$$

(24)

Given a cumulative cut-set sequence $\mathcal{W}$ and a permutation sequence $\mathcal{T}$ for $\mathcal{W}$, we can distribute the source information $Y_i$ over the edges in $C_i$ as follows:

$$I(Y_i; U_{C_i}|Y_{1:i-1}) = \sum_{e \in C_i} I(Y_i; U_e|Y_{1:i-1}, U_{T_i(e)})$$

(25)

Without loss of generality, let $\mathcal{W}(e) = \{C_{n_1}, \ldots, C_{n_k}\}$, where $1 \leq n_1 < \cdots < n_k \leq K$. The second feature is presented below.

**Definition 5.** Given a cut-set sequence $\mathcal{W}$, we say that it is distributive if there exists a permutation sequence $\mathcal{T}$ for $\mathcal{W}$ such that for each $e \in \bigcup_{i=1}^{K} C_i$, the following conditions are satisfied: for all $1 \leq j \leq K$,

$$\alpha(e') \leq n_k \quad \forall e' \in T_{n_j+1}(e) - T_{n_j}(e)$$

(26)

$$\alpha(e') \leq n_{j+1} - 1 \quad \forall e' \in T_{n_j}(e) - T_{n_{j+1}}(e)$$

As shown in Example 2, let $T_1 = (e_1, e_2, e_3)$, and $T_2 = (e_2, e_3)$. For $e_3$, $W(e_3) = \{C_1, C_2\}$, $T_2(e_3) - T_1(e_3) = \emptyset$, and thus, (26) is trivially satisfied; $T_1(e_3) - T_2(e_3) = \{e_1\}$, $\alpha(e_1) = 1$, and (27) is satisfied. Similarly, we can verify other edges. Hence, $\mathcal{W}$ is distributive.

The above two features ensure that the information from multiple unicast sessions that is distributed on an edge $e \in \bigcup_{i=1}^{K} C_i$ can be completely obtained from $U_e$.

**Lemma 2.** Consider a network code as defined in Definition 1. Given a cumulative cut-set sequence $\mathcal{W}$, if $\mathcal{W}$ is distributive, for each $e \in \bigcup_{i=1}^{K} C_i$, the following inequality holds:

$$\sum_{1 \leq i \leq K, e \in C_i} I(Y_i; U_e|Y_{1:i-1}, U_{T_i(e)}) \leq H(U_e)$$

(28)

The third feature is presented below.

**Definition 6.** Given a path-set sequence $K$ for $\mathcal{W}$, we say that $\mathcal{K}$ is extendable, if for all $1 \leq i < j \leq K$, $P_i \cap P_j = P_2 \cap P_j$ such that $P_1$ overlaps with $P_2$, $P_1 \cap C_i = P_2 \cap C_j$. As shown in Example 2, let $K = \{P_1, P_2\}$. Clearly, we have $P_{12} \cap P_2 = \{e_2, e_3\}$, $P_{13} \cap C_1 = P_{21} \cap C_2 = \{e_2\}$, and $P_{13} \cap P_{22} = \{e_3\}$, $P_{13} \cap C_1 = P_{22} \cap C_2 = \{e_3\}$. Thus, $\mathcal{K}$ is extendable.

**Definition 7.** A network with multiple unicast sessions is said to be information-distributive, if there exist a cumulative and distributive cut-set sequence $\mathcal{W}$, and an extendable path-set sequence $K$ for $\mathcal{W}$ in the network.

As shown in the next theorem, the three features together guarantee that the network is routing-optimal.

**Theorem 1.** If a network is information-distributive, it is routing-optimal.

**Example 3.** Consider the network shown in Fig. 1b. Define the following cut-sets: $C_1 = \{(s_1, v_1), (v_2, v_3), (v_3, v_5)\}$, $C_2 = \{(v_2, v_3), (v_4, v_5)\}$, $C_3 = \{(v_6, v_7), (s_3, d_3)\}$, and let $W = \{C_1\}_i^3$. Define the following paths: $P_{11} = \{(s_1, v_1), (v_1, d_1)\}$, $P_{12} = \{(s_1, v_2), (v_2, v_3), (v_3, d_1)\}$, $P_{13} = \{(s_1, v_4), (v_4, v_5), (v_5, d_1)\}$, $P_{21} = \{(s_2, v_2), (v_2, v_3), (v_3, d_2)\}$, $P_{31} = \{(s_3, v_6), (v_6, v_7), (v_7, d_3)\}$, $P_{33} = \{(s_3, d_3)\}$, and let $K = \{P_{11}, P_{12}, P_{13}, P_{21}, P_{31}, P_{33}\}$. It can be verified that $W$ is cumulative and distributive, and $K$ is extendable. The network is information-distributive.

**C. The Converse Is Not True**

Note that information-distributive networks don’t subsume all possible routing-optimal networks. In Fig. 2, we show a routing-optimal network that is not information-distributive. For detailed discussion, see [5].

**IV. MORE EXAMPLES**

We defer proofs to [5].

**A. Index Coding**

We consider the index coding problem described by [4]. In this problem, there are $K$ terminals $t_1, \ldots, t_K$, a broadcast station $s$, and $K$ source messages $X_1, \ldots, X_K$, all available at $s$. All $X_i$’s are mutually independent random variables uniformly distributed over alphabet $\mathcal{X}_i = \{1, \ldots, 2^m\}$. Each terminal requires $X_i$, and has acquired a subset of source messages $\mathcal{H}_i$ such that $X_i \notin \mathcal{H}_i$. $s$ uses an encoding function $\phi : \prod_{i=1}^{K} \mathcal{X}_i \rightarrow \{1, \ldots, 2^l\}$ to encode the source messages, and broadcasts the encoded message to the terminals through an error-free broadcast channel. Each $t_i$ uses a decoding function $\psi_i$ to decode $X_i$ by using the received message and the messages in $\mathcal{H}_i$. The encoding function $\phi$ and the decoding functions $\psi_i$’s are collectively called an index code, and $l$ is the length of this index code. The minimum length of an index code is denoted by $l_{\text{min}}$.

This index coding problem can be cast to a multiple-unicast network coding problem over a network $G_1 = (V_1, E_1)$, where $V_1 = \{s_1, d_i : 1 \leq i \leq K\} \cup \{u, v\}$, $E_1 = \{(s_i, u), (u, v), (v, d_i) : 1 \leq i \leq K\} \cup \{(u, v)\} \cup \{(s_i, d_i) : X_i \notin \mathcal{H}_i\}$. The $K$ unicast sessions are $(s_1, d_1), \ldots, (s_K, d_K)$. It can be verified that there exists an index code of length $l_i$, if and only if $R = \{l_1, \ldots, l_K\}$ is achievable by network coding in $G_1$.

Let $C_i = \{(u, v), P_i, C_{i1}, (s_i, u), (u, v), (v, d_i)\}$. Define $\mathcal{W} = \{C_i\}_{i=1}^{K}$ and $K = \{P_i\}_{i=1}^{K}$, where $P_i = \{P_i\}$. Since each $C_i$ contains only one edge, $\mathcal{W}$ is distributive. Meanwhile, since all $P_i$’s overlap at $(u, v)$, $K$ is extendable.

The following theorem states that if the optimal solution to the index coding problem is to let the broadcast station transmit raw packet, i.e., no coding is needed, then the corresponding multiple-unicast network is information-distributive, and the converse is also true.

**Theorem 2.** $l_{\text{min}} = mK$ if and only if $\mathcal{W}$ is cumulative, i.e., $G_1$ is information-distributive.
B. Single Unicast with Hard Deadline Constraint

In this example, we consider the network coding problem for a single-unicast session \((s, d)\) over a network \(G = (V, E)\), where each edge \(e\) is associated with a delay \(d_e \in \mathbb{Z}_{>0}\), and each node has a memory to hold received data. Given a directed path \(P\), let \(d(P) = \sum_{e \in P} d_e\) denote its delay. For \(e \in E\), let \(\delta(e)\) denote the minimum delay of directed paths from \(s\) to \(t\) (\(e\)). The data transmission in the network proceeds in time slots. The messages transmitted from \(s\) is represented by a sequence \(Y[t]\), where \(Y[t]\) is a uniformly distributed random variable, and represents the message transmitted from \(s\) at time slot \(t\). All \(Y[t]\)’s are mutually independent. We require that each \(Y[t]\) must be received by \(d\) within \(\tau\) time slots. Otherwise, it is regarded as useless, and is discarded. This problem was first proposed by [7] [8]. Recently, it has been shown that network coding can improve throughput by utilizing over-delayed information [9].

This problem can be cast to an equivalent network coding problem for multiple unicast sessions. We construct a timeextended graph \(\tilde{G} = (V', E)\) as follows: the node set is \(V = \{s_0, d_0\} \cup \{i[t] : 0 \leq t \leq K + \tau\}\); for each \(e = (u, v) \in E\) and \(0 \leq t \leq K + \tau - d_e\), we add an edge \(e[t] = (u[t], v[t] + d)\) to \(\tilde{E}\); for \(u \in V\) and \(0 \leq t \leq K + \tau - 1\), we add \(M\) edges from \(u[t]\) to \(u[t+1]\), where \(M\) is the amount of memory available at \(u\); for each \(0 \leq t < K\), we add \(J\) edges from \(s_j\) to \(s[t]\) and \(J\) edges from \(d[t+\tau]\) to \(d_t\), where \(J\) is a sufficiently large integer. Thus, the original single unicast session \((s, d)\) is cast to \(K + 1\) unicast sessions \((s_0, d_0), \ldots, (s_K, d_K)\) over \(\tilde{G}\).

Let \(\tilde{G}[t]\) denote the routing domain for \((s_i, d_0)\), and \(m = \text{min}(s_0, d_0, \tilde{G}[0])\). It can be seen that each \(\tilde{G}[t]\) is simply a time-shifted version of \(\tilde{G}[0]\). Given a subset of edges \(U \subseteq \tilde{E}\), define \(U[t] = \{(u[k + t], v[t + l]) : (u[k], v[l]) \in U\}\). Let \(C = \{e[j][t] : 1 \leq j \leq m\}\) be a cut-set between \(s_0\) and \(d_0\) such that \(e_j \in E\), for \(1 \leq j \leq m\), and \(P = \{P_1, \ldots, P_m\}\) a set of edge disjoint paths from \(s_0\) to \(d_0\) such that \(e_j[e] \in P_j\) for \(1 \leq j \leq m\). Let \(P[t] = \{P[t] : P \in P\}\). We consider the cut-set sequence \(W = (C[t]_{t=1}^K)\), and the path-set sequence \(\mathcal{K} = (P[t])_{t=0}^K\).

**Lemma 3.** \(W\) is cumulative.

Given \(U \subseteq \tilde{E}\), a recurrent sequence of \(U\) is a sequence consisting of all the edges in \(U\) that are time-shifted versions of the same edge. \(C[0]\) is said to be distributive if there is a re-indexing of the edges in \(C[0]\) such that for each recurrent sequence \((e_p[t_n])_{n=1}^t\) of \(C[0]\), the following conditions are satisfied:

1) for each \(1 < j \leq k\), if \(e_q[t_q] \in C[0]\) lies before \(e_p[t_n]\), and \(e_p[t_q] - e_q[t_q] - t_{n+1} - 1\) \(\not\in C[0]\), then \(t_q - \delta(e_q) \leq t_{n+1} - t_{n+1} - 1\);

2) for each \(1 \leq j < k\), if \(e_q[t_q] \in C[0]\) lies before \(e_p[t_n]\), and \(e_p[t_q] + t_{n+1} - t_{n+1} \not\in C[0]\), then \(t_q - \delta(e_q) \leq t_{n+1} - t_{n+1}\).

\(P\) is said to be extendable if for all \(P_1, P_2 \in \mathcal{P}\) and \(e[k], e[l] \in \tilde{E}\) such that \(e[k] \in P_1\) and \(e[l] \in P_2\), \(e_1 = e_2\) and \(t_1 - t_2 = k - l\).

**Theorem 3.** If \(C[0]\) is distributive, and \(P\) is extendable, \(\tilde{G}\) is information-distributive, and thus is routing-optimal.

**Example 4.** In Fig. 3a, we show an example of single unicast with delay constraint \(\tau = 7\). In Fig. 3b, we show the routing domain \(\tilde{G}[t]\) for \((s_0, d_0)\). Let \(C[0]\) = \{\(e_5[5], e_6[2], e_8[6]\)\}, and \(P = \{P_1, P_2, P_3\}\), where \(P_1, P_2, P_3\) are marked as black dashed lines in Fig. 3b. It can be verified that \(C[0]\) is distributive, and \(P\) is extendable.

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**REFERENCES**


